Focused one-cycle electromagnetic pulses

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We describe several families of exact unbounded solutions of Maxwell's equations in vacuum. These solutions depict one- (or $1\frac{1}{2}$-) cycle electromagnetic pulses whose fields are of either transverse magnetic or transverse electric character and are confined to toroidal wave packets that converge to a focus and then diverge in a manner that is expected from familiar rules of diffraction. These ''focused doughnut'' pulses constitute a subset of the ''modified power spectrum'' pulse solutions discovered by Ziolkowski [Phys. Rev. A **39**, 2005 (1989)]. We derive the total energy, the energy spectrum, the ability to accelerate an electron, and other properties of these focused doughnut pulse solutions. [S1063-651X(96)07407-7]

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I. INTRODUCTION

Ziolkowski has discovered a class of exact, finite-energy solutions of Maxwell's equations (in vacuum), which he has named ''electromagnetic directed-energy pulse trains'' or EDEPT solutions [1]. These finite-energy solutions for the electric and magnetic field vectors in vacuum are derived from a scalar complex potential $f(\vec{r},t)$ that involves the Laplace transform of essentially any function $F(k)$ that has the property that $\int_0^{+\infty} dk|F(k)|^2 k^m$ is finite for $m=1$ and 2. Using a particular $F(k)$ having a known Laplace transform, Ziolkowski found a particularly simple subset of the EDEPT solutions that was parametrized by four real positive numbers, which he named ''modified power spectrum'' pulse solutions or simply MPS pulses [1]. Ziolkowski explored in detail the MPS pulse solutions in a certain region of parameter space in which he found these solutions to depict efficient directed electromagnetic energy transfer in space [1]. Here we explore an entirely different region of the parameter space of MPS solutions where they depict a family of focused pulses that are essentially one cycle long. We call these pulses ''focused doughnut'' pulses as their energy distribution in space resembles a doughnut that focuses to some minimum diameter d_m and then diffracts beyond the focal region with the same relation between d_m, the wavelength, and the far-field divergence angle as for normal optical beams.

In Sec. II we review Ziolkowski's general expression for the complex Hertz potential $\hat{\mathbf{z}}f(\vec{r},t)$ whose real and imaginary parts yield separate solutions for the electric and magnetic vectors $\vec{E}$ and $\vec{H}$ of the focused doughnut pulses. The imaginary part yields a pulse that is one optical cycle long (in a sense we specify) and we call it the ''short-pulse solution.'' The real part yields a pulse that is $1\frac{1}{2}$ optical cycles long and we call it the ''long-pulse solution.'' Both solutions have their electric vectors always transverse to the direction of propagation z and so we call these ''TE'' solutions. In vacuum, any solution generates another dual solution under the transformation $\vec{\mathbf{E}}_{\rm TE} \to -\vec{\mathbf{H}}_{\rm TM}$ and $\vec{\mathbf{H}}_{\rm TE} \to \vec{\mathbf{E}}_{\rm TM}$. We obtain thereby a third and a fourth solution whose magnetic vectors are transverse, and which we call ''TM'' short- and long-pulse solutions.

In Sec. III we derive the energy of MPS (and of all EDEPT) solutions of Maxwell's equations. We find a result that is qualitatively different from Ziolkowski's [1] and explain the difference.

In Sec. IV we derive the frequency spectrum of the pulse energy for each of the quartet of focused-doughnut solutions. We find that each has the same simple bell-shaped energy spectrum, which, as a function of angular frequency ω, is proportional to ω^4 at small ω and to $\omega^2\exp(-T\omega)$ at large ω, where T is the length of the pulse.

In Sec. V we calculate the ability of a TM focused doughnut pulse to accelerate a copropagating electron. That is, we calculate the energy W given by a TM focused doughnut pulse of energy $U_{\rm TM}$ to a copropagating relativistic electron. We find W to be of the order of the geometric mean of the pulse energy $U_{\rm TM}$ and the Coulomb energy $U_{\rm Coul}$ of two electrons separated by the pulse wavelength. A 1-J pulse of 1-μm wavelength imparts nearly 1 GeV of energy to the copropagating electron.

In our concluding section (Sec. VI) we note a few of the many remaining unanswered questions about focused doughnut pulses in particular and MPS pulses in general.

II. FOCUSED DOUGHNUT PULSE FIELDS

To obtain the simple expressions for the focused doughnut electric and magnetic fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$ we use Hertz's method. First we find an appropriate scalar generating function $f(\mathbf{r},t)$ that satisfies Helmholtz's wave equation in vacuum

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) f(\mathbf{r},t) = 0, \tag{1}$$

where c is the velocity of light.

Next we construct a vector potential $\mathbf{A}(\mathbf{r},t)$ in the Coulomb gauge (in which both $\nabla \cdot \mathbf{A}$ and the scalar potential are zero) by

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$$\mathbf{A}(\mathbf{r},t)=\mu_0\,\mathrm{curl}\hat{\mathbf{z}}f(\mathbf{r},t), \tag{2}$$

where $\hat{\mathbf{z}}$ is a unit vector in the direction of pulse propagation. We use SI units throughout, in which the permeability of the vacuum $\mu_0 = 4\pi\times10^{-9}$ J s^2 C^{-2} m^{-1}. Therefore, the scalar generating function f has units Am.

Using cylindrical coordinates ρ, θ, and z (such that $x=\rho\cos\theta$, $y=\rho\sin\theta$, and $z=z$) we have from (2) that both the real and imaginary parts of the following transverse electric (TE) fields satisfy Maxwell's equations in vacuum separately, because we assume that f is independent of θ:

$$E_\theta=\mu_0\partial_\rho\partial_t f, \tag{3}$$

$$H_\rho=\partial_\rho\partial_z f, \tag{4}$$

$$H_z=\left(\partial_z^2-\frac{1}{c^2}\partial_t^2\right)f, \tag{5}$$

where

$$\partial_z f\equiv\partial f/\partial z,\quad \partial_\rho f\equiv\partial f/\partial\rho,\quad \partial_t f\equiv\partial f/\partial t. \tag{6}$$

We will symbolize the two real (long- and short-pulse) solutions thus obtained by $(\mathbf{E}'_{\mathrm{TE}},\mathbf{H}'_{\mathrm{TE}})$ and $(\mathbf{E}''_{\mathrm{TE}},\mathbf{H}''_{\mathrm{TE}})$, respectively, and abbreviate $\mathbf{E}_{\mathrm{TE}}=\mathbf{E}'_{\mathrm{TE}}+i\mathbf{E}''_{\mathrm{TE}}$, etc., for $\mathbf{H}_{\mathrm{TE}}$.

It is evident that since we are solving Maxwell's equations in vacuum we obtain two additional real TM solutions that have transverse magnetic field and whose field pairs we write $(\mathbf{E}'_{\mathrm{TM}},\mathbf{H}'_{\mathrm{TM}})$ and $(\mathbf{E}''_{\mathrm{TM}},\mathbf{H}''_{\mathrm{TM}})$. We write $\mathbf{E}_{\mathrm{TM}}=\mathbf{E}'_{\mathrm{TM}}+i\mathbf{E}''_{\mathrm{TM}}$, etc., for $\mathbf{H}_{\mathrm{TM}}$. The TM solutions come directly from the TE solutions by

$$\mathbf{E}_{\mathrm{TM}}=\sqrt{\frac{\mu_0}{\epsilon_0}}\mathbf{H}_{\mathrm{TE}},$$

$$\mathbf{H}_{\mathrm{TM}}=-\sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{E}_{\mathrm{TE}}. \tag{7}$$

We will use the MPS solution of Ref. [1] discovered by Ziolkowski:

$$f=f_0\frac{e^{-s/q_3}}{(q_1+i\tau)(s+q_2)^\alpha}. \tag{8}$$

Here q_1, q_2, q_3 are real positive adjustable parameters with units of length. The real dimensionless parameter α must satisfy $\alpha\geqslant1$ in order for the electromagnetic pulse of (3) to (5) have finite energy. We will assume (without loss of generality) that f_0 is a real constant. Other trivial parameters can be introduced that only change the origin of coordinates. In (8) we have used Ziolkowski's abbreviations

$$s\equiv\rho^2/(q_1+i\tau)-i\sigma,\quad \rho^2\equiv x^2+y^2,\quad \tau\equiv z-ct,\quad \sigma\equiv z+ct. \tag{9}$$

For convenience, we have reduced the five redundant parameters a, α, b, β, and z_0 used by Ziolkowski in his expression for f [Eq. (3.4) of Ref. [1]] to the four independent parameters $q_1\equiv z_0$, $q_2\equiv a\beta$, $q_3\equiv\beta/b$, and α. Ziolkowski performed extensive numerical and graphical studies of MPS pulses having $q_1\ll q_3\ll q_2$ and $\alpha=1$. Here we study the qualitatively different solutions having $q_1\leqslant q_2$, $\alpha=1$, and $q_3\rightarrow\infty$. This yields the two-parameter subset of the MPS solutions, which we have called "focused doughnut" pulses. We find that q_1 is a measure of the wavelength of these essentially one-cycle pulses and q_2 is a measure of the "Rayleigh range" or depth of the focal region.

Substituting (8) in (3)–(5) with $1/q_3=0$ and $\alpha=1$, we find

$$E_\theta=-4if_0\sqrt{\frac{\mu_0}{\epsilon_0}}\rho(q_1+q_2-2ict)/[\rho^2+(q_1+i\tau)(q_2-i\sigma)]^3, \tag{10}$$

$$H_\rho=4if_0\rho(q_2-q_1-2iz)/[\rho^2+(q_1+i\tau)(q_2-i\sigma)]^3, \tag{11}$$

$$H_z=-4f_0[\rho^2-(q_1+i\tau)(q_2-i\sigma)]/[\rho^2+(q_1+i\tau)(q_2-i\sigma)]^3. \tag{12}$$

We note several characteristics of these focused doughnut solutions in the "well-collimated" or "weakly focused" limit where the Rayleigh length q_2 is much larger than the wavelength q_1. First, an outgoing weakly focused (TE or TM) pulse described by the imaginary parts of (10)–(12) has a field on axis, at $\rho=0$ and $z\gg q_1$, that crosses zero once in time as the pulse goes by. Therefore we call this pair of solutions $(E''_{\mathrm{TE}},\ H''_{\mathrm{TE}})$ and $(E''_{\mathrm{TM}},\ H''_{\mathrm{TM}})$ "one-cycle" or "short" pulses. The corresponding on-axis field of the real parts of (12) crosses zero twice in time as the pulse goes by. Therefore we call the pair of solutions $(E'_{\mathrm{TE}},\ H'_{\mathrm{TE}})$ and $(E'_{\mathrm{TM}},\ H'_{\mathrm{TM}})$ "$1\frac{1}{2}$-cycle" or "long" pulses.

Second, we note that, at $t=0$, when the maximum focus occurs in the region around $\vec{r}=0$, the energy density $(\mu_0H^2+\epsilon_0E^2)/2$ of a short focused doughnut solution decreases with large distance r from the origin as r^{-10}. ($r^2\equiv z^2+\rho^2$.) However, the energy density of a long focused doughnut pulse decreases with large distance r only as r^{-8}. In the next section we will find that each of the four solutions (having electric fields $\mathbf{E}'_{\rm TE}$, $\mathbf{E}''_{\rm TE}$, $\mathbf{E}'_{\rm TM}$, and $\mathbf{E}''_{\rm TM}$ as we have defined them along with their accompanying magnetic fields) has the same energy.

III. ENERGY OF EDEPT SOLUTIONS

Following Ziolkowski [1], we will calculate the total energy in the broader class of transverse electric EDEPT solutions for which the generating function f to be used in (3)–(5) is given by

$$f=\frac{f_0}{q_1+i\tau}\int_0^{+\infty}dkF(k)e^{-ks}. \tag{13}$$

Because we obtain a result somewhat different from Ziolkowski's, we retrace his derivation in enough detail to show the reader when various differences arise and to encourage the reader to check our calculation. The situation here is unusual in that neither Ziolkowski nor we had any well-established result with which to compare our energy expression, either in some limit of parameter space or in its functional dependence on the four parameters in (8).

The proper form of $F(k)$ to use to obtain the MPS function (8) is given in Eq. (3.3) of Ref. [1]. Note that the $F(k)$ in our (13) equals that of Eq. (2.6) of Ref. [1] but multiplied by $4\pi i f_0$. The electromagnetic pulse derived from $F(k)=\delta(k-k_0)$ was studied by Brittingham, who called this solution a focus wave mode [2]. Like a plane wave, the focus wave mode has finite energy density but infinite total field energy.

We will call the energy of the real part of the complex solutions E_θ, H_ρ, H_z of (3)–(5) U_+ and we call the energy of the imaginary part U_-. Using the Poynting formula for the energy in SI units, we may combine expressions for these two energies as

$$U_\pm=\frac{1}{8}\int_{-\infty}^{+\infty}dz\int_0^{+\infty}\rho\, d\rho\int_0^{2\pi}d\theta[\pm\epsilon_0(E_\theta\pm E_\theta^*)^2 \pm\mu_0(H_\rho\pm H_\rho^*)^2\pm\mu_0(H_z\pm H_z^*)^2]. \tag{14}$$

The fields do not have any dependence on θ so that the θ integration gives a factor 2π. We find that the energies of the two solutions can be written as the sum of two integrals

$$U_\pm=\pm(I_1+I_1^*)+I_2, \tag{15}$$

with

$$I_1=\frac{\pi}{4}\int_{-\infty}^{+\infty}dz\int_0^{+\infty}\rho\, d\rho(\epsilon_0E_\theta^2+\mu_0H_\rho^2+\mu_0H_z^2) \tag{16}$$

and

$$I_2=\frac{\pi}{2}\int_{-\infty}^{+\infty}dz\int_0^{+\infty}\rho\, d\rho(\epsilon_0|E_\theta|^2+\mu_0|H_\rho|^2+\mu_0|H_z|^2). \tag{17}$$

Ziolkowski omitted I_1 and I_1^* in his formula for the energy [1], but, as we shall show below, this leads to no fundamental error because I_1 is found to be equal to zero. Note that the starting energy expression of Ref. [1] (B8) is our I_2 multiplied by 4.

Following Ziolkowski, we now use the following decomposition to calculate the two integrals I_1 and I_2:

$$E_\theta=f_0\int_0^{+\infty}dkF(k)\tilde{E}_\theta,$$

$$H_\rho=f_0\int_0^{+\infty}dkF(k)\tilde{H}_\rho, \tag{18}$$

$$H_z=f_0\int_0^{+\infty}dkF(k)\tilde{H}_z.$$

Here $\tilde{E}_\theta$, $\tilde{H}_\rho$, and $\tilde{H}_z$ are the complex solutions to Maxwell's equations constructed using $f=e^{-ks}/(q_1+i\tau)$ in (3)–(5):

$$\tilde{E}_\theta=\sqrt{\frac{\mu_0}{\epsilon_0}}\left(-\frac{4ik\rho}{(q_1+i\tau)^2}+\frac{2ik^2\rho^3}{(q_1+i\tau)^3}-\frac{2ik^2\rho}{(q_1+i\tau)}\right)\frac{e^{-ks}}{q_1+i\tau}, \tag{19}$$

$$\tilde{H}_\rho=\left(\frac{4ik\rho}{(q_1+i\tau)^2}-\frac{2ik^2\rho^3}{(q_1+i\tau)^3}-\frac{2ik^2\rho}{(q_1+i\tau)}\right)\frac{e^{-ks}}{q_1+i\tau}, \tag{20}$$

$$\tilde{H}_z=\left(\frac{4k}{(q_1+i\tau)}-\frac{4k^2\rho^2}{(q_1+i\tau)^2}\right)\frac{e^{-ks}}{q_1+i\tau}. \tag{21}$$

We substitute Eqs. (18)–(21) into (16) to get

$$I_1=2\pi\mu_0f_0^2\int_0^{+\infty}dk\int_0^{+\infty}dk'F(k)F(k')e^{-(k+k')s} \times\int_{-\infty}^{+\infty}dz\int_0^{+\infty}d\rho kk'(q_1+i\tau)^{-4} \times\left(-\frac{4\rho^3}{(q_1+i\tau)^2}-\frac{kk'\rho^7}{(q_1+i\tau)^4}-kk'\rho^3+\frac{2k'\rho^5}{(q_1+i\tau)^3}+\frac{2k\rho^5}{(q_1+i\tau)^3}+2\rho+\frac{2kk'\rho^5}{(q_1+i\tau)^2}-\frac{2k'\rho^3}{(q_1+i\tau)}-\frac{2k\rho^3}{(q_1+i\tau)}\right) \tag{22}$$

We use the relation $\int_0^{+\infty}dx x^{2m+1}e^{-ax^2}=m!/2a^{m+1}$, $\mathrm{Re}(a)>0$ with $x=\rho$ and $a=(k+k')/(q_1+i\tau)$ to perform the ρ integration. After some algebraic manipulations, we get

$$I_1=\pi\mu_0 f_0^2\int_0^{+\infty}dk\int_0^{+\infty}dk' F(k)F(k')k^2k'^2 \times\int_{-\infty}^{+\infty}e^{+i(k+k')\sigma}dz\frac{1}{(q_1+i\tau)^2(k+k')^2} \times\left(\frac{4}{(q_1+i\tau)(k+k')}-\frac{6}{(q_1+i\tau)^2(k+k')^2}-1\right). \quad (23)$$

Because q_3 is real, k and k' are to be integrated along the real k axis. In this case, we may use the integral formulas

$$\int_{-\infty}^{+\infty}\frac{e^{ix}}{(a+ix)^2}dx=2\pi e^{-a},$$

$$\int_{-\infty}^{+\infty}\frac{e^{ix}}{(a+ix)^3}dx=\pi e^{-a}, \quad (24)$$

$$\int_{-\infty}^{+\infty}\frac{e^{ix}}{(a+ix)^4}dx=\frac{\pi}{3}e^{-a}.$$

With the substitutions $x=(k+k')z$ and $a=(k+k')(q_1-ict)$, one sees immediately that the z integral in Eq. (23) vanishes because the three terms in the parentheses give relative contributions 2, -1, -1. This means that the total energies U_+ and U_- of Eq. (15) are both given by I_2. That is, the energies contained in both the real ($\mathbf{E}'$, $\mathbf{H}'$) and the imaginary ($\mathbf{E}''$, $\mathbf{H}''$) part of the fields $\mathbf{E}$ and $\mathbf{H}$ are given by the integral I_2 , as are the energies of both dual solutions constructed by the transformations in (7). Following Ziolkowski, we call this universal energy function U_{EM} and summarize this result by

$$U_{\mathrm{EM}}=I_2=U_+=U_-. \quad (25)$$

.

The calculation of the second integral I_2 is quite similar to that of I_1. By substituting Eqs. (18)–(21) into (17), we find

$$I_2=4\pi\mu_0|f_0|^2\int_0^{+\infty}dk\int_0^{+\infty}dk' F(k)F^*(k') \times\int_{-\infty}^{+\infty}dz e^{-i(k'-k)\sigma} \times\int_0^{+\infty}d\rho e^{-[(k+k')q_1+i(k'-k)\tau]\rho^2/(q_1^2+\tau^2)} \times\left(\frac{k^2k'^2\rho^7}{(q_1^2+\tau^2)^4}+\frac{2k^2k'^2\rho^5}{(q_1^2+\tau^2)^3}+\frac{k^2k'^2\rho^3}{(q_1^2+\tau^2)^2}+\frac{4kk'\rho^3}{(q_1^2+\tau^2)^3} -\frac{2kk'^2\rho^5}{(q_1^2+\tau^2)^3(q_1-i\tau)}-\frac{2k^2k'\rho^5}{(q_1^2+\tau^2)^3(q_1+i\tau)} +\frac{2kk'\rho}{(q_1^2+\tau^2)^2}-\frac{2kk'^2\rho^3}{(q_1^2+\tau^2)^2(q_1-i\tau)} -\frac{2k^2k'\rho^3}{(q_1^2+\tau^2)^2(q_1+i\tau)}\right). \quad (26)$$

This result is similar to that of Ziolkowski [1] at the top of p. 2032 of Ref. [1], except for the factors in front of the integral. This is explained because we used slightly different definitions for f, $F(k)$, $\widetilde{E}_\theta$, $\widetilde{H}_\rho$, and $\widetilde{H}_z$ and Ziolkowski omitted a factor $\mu_0/4$ throughout his calculations.

We next perform the ρ integration and find

$$I_2=2\pi\mu_0 f_0^2\int_0^{+\infty}dk\int_0^{+\infty}dk' F(k)F^*(k')k^2k'^2 \times\int_{-\infty}^{+\infty}dz e^{-i(k'-k)\sigma}\left(\frac{6}{[(k+k')q_1+i(k'-k)\tau]^4} +\frac{4}{[(k+k')q_1+i(k'-k)\tau]^3} +\frac{1}{[(k+k')q_1+i(k'-k)\tau]^2}\right). \quad (27)$$

Following Ziolkowski, we perform the z integration, using the following expression, which he derived for Λ real and positive:

$$\int_{-\infty}^{+\infty}dy\frac{e^{-ixy}}{(\Lambda+ixy)^m}=2\pi\delta(x)\frac{e^{\Lambda}}{\Lambda^{m-1}}E_m(\Lambda). \quad (28)$$

Here

$$E_m(\Lambda)\equiv\int_\Lambda^{+\infty}\frac{e^{-\Lambda t}}{t^m}dt. \quad (29)$$

With $x=k'-k$ and $\Lambda=(k+k')q_1$, we obtain, following Ziolkowski, the integral expression

$$I_2=4\pi^2\mu_0 f_0^2\int_0^{+\infty}dk|F(k)|^2k^4e^{2kq_1}\left(\frac{6E_4(2kq_1)}{(2kq_1)^3} +\frac{4E_3(2kq_1)}{(2kq_1)^2}+\frac{E_2(2kq_1)}{2kq_1}\right). \quad (30)$$

With the recurrence relation $E_{n+1}(x)=(1/n)[e^{-x}-xE_n(x)]$, this simplifies to

$$I_2=(\pi^2\mu_0 f_0^2/q_1^4)\int_0^{+\infty}dk|F(k)|^2[kq_1+(kq_1)^2]. \quad (31)$$

This is the form that is useful for calculating the energy of any finite-energy EDEPT solution of Maxwell's equation in vacuum.

We now use the general result (31) to calculate the energy of the three-parameter MPS solution generated by using the function $f(\mathbf{r},t)$ of (8) (in the case $\alpha=1$) in the field relations (3)–(5). The focused doughnut solutions are included in this class, which has

$$F(k)=\begin{cases} e^{-q_2(k-1/q_3)} & \text{if } k>1/q_3 \\ 0 & \text{if } k\leqslant 1/q_3. \end{cases} \quad (32)$$

Substituting Eq. (32) for $F(k)$ into (31) gives, after integration

$$U_{\rm EM}=\pi^2\mu_0 f_0^2\frac{[(1+q_2/q_1)(1+2q_2/q_3)+2(q_2/q_3)^2]}{4q_1^2q_2^3}. \tag{33}$$

Ziolkowski's expression (B11) in Ref. [1] for this result is quite different. The $1+q_2/q_1$ in our parentheses is replaced by unity. The 4 in our denominator is omitted and our μ_0/q_1^2 is replaced by $1/q_1^4$. Ziolkowski carries this error over to his Eqs. (3.18) and (3.18′).

To check the result (33), we integrated (14) numerically for TE pulses with (i) $q_1=q_2=1$, $q_3\to+\infty$, and $ct=0$; (ii) $q_1=0.01$, $q_2=1$, $q_3\to+\infty$, $ct=0$, and $ct=10$; and (iii) $q_1=0.01$, $q_2=1$, $q_3=10$, and $ct=0$. We obtained the same number as from (33) to within 0.5%. However, we believe that the derivation of the results (31) and (33) is sufficiently complicated that the results cannot be said to be well established until confirmed by others.

IV. ENERGY SPECTRA OF FOCUSED DOUGHNUT PULSES

If the focused doughnut solutions are to be compared with other types of electromagnetic pulses in, for example, their ability to accelerate an electron, their energy spectra must be known. Here we derive the energy spectra of the quartet of focused doughnut solutions, deriving first the spectrum V'_ω of the long TE pulse whose energy U_+ [given by (33) with $q_3^{-1}=0$] is related to V'_ω in the usual way:

$$U_+=\int_0^\infty\frac{d\omega}{2\pi}V'_\omega. \tag{34}$$

To calculate the spectrum V'_ω we imagine a very large sphere covered with spectrometers that measure the energy spectrum $v'_\omega(\mathbf{r})$ of the light passing outward through a unit area at each point of the sphere. Then

$$V'_\omega=\int_S d^2r v'_\omega(\mathbf{r}), \tag{35}$$

where S indicates the surface of a sphere in the far field of the outward traveling pulse. We represent the θ component E'_θ of the long TE pulse in the far field by E'_f. Similarly we write the θ component E''_θ of the short TE pulse in the far field E''_f and define E_f by

$$E_f\equiv E'_f+iE''_f. \tag{36}$$

The complex electric field E_f is, from (10), the limit at large radius $r\equiv(\rho^2+z^2)^{1/2}$ of $\mu_0\partial^2 f(\mathbf{r},t)/\partial\rho\partial t$. We define the (time) Fourier transform

$$F_\omega(\mathbf{r})\equiv\int_{-\infty}^{+\infty}dte^{i\omega t}E_f(\mathbf{r},t). \tag{37}$$

The far-field Fourier transforms F'_ω and F''_ω of E'_f and E''_f, respectively, are then

$$F'_\omega=(F_\omega+F^*_{-\omega})/2 \tag{38}$$

and

$$F''_\omega=(F_\omega-F^*_{-\omega})/2i. \tag{39}$$

In terms of these, the energy spectrum V'_ω of the long TE pulse is given by using

$$v'_\omega(\mathbf{r})=2c\epsilon_0|F'_\omega(\mathbf{r})|^2, \tag{40}$$

where ω is positive, and similarly for the spectrum v''_ω of the short TE pulse.

In the far field of the outgoing focused doughnut pulse (where $r\gg q_2$)

$$f\to-\frac{f_0}{r}\frac{1}{(ct-r+iQ)}, \tag{41}$$

where

$$Q=\tfrac{1}{2}[q_2+q_1-(q_2-q_1)\cos\psi]. \tag{42}$$

Here $\cos\psi\equiv z/r$ and ψ is the polar angle in the spherical coordinates $(r,\ \psi,\ \theta)$. The Fourier transform (37) becomes a simple contour integral with a single pole and yields

$$F(\omega)=\begin{cases}\left(\dfrac{-\pi\mu_0 f_0\omega}{c}\right)\dfrac{\partial}{\partial\rho}\left(\dfrac{\exp(i\omega r/c+\omega Q/c)}{r}\right), & \omega<0\\ 0, \quad \omega\geq 0.\end{cases} \tag{43}$$

The only term from the derivative $\partial/\partial\rho$ that contributes in the far field involves the derivation of $i\omega r/c$ in the exponent. Therefore the derivative may be replaced by the factor $(i\omega c^{-1}\sin\psi)$ to give

$$F'_\omega=\left(\frac{i\pi\mu_0 f_0\omega|\omega|\sin\psi}{2rc^2}\right)\exp(i\omega r/c-|\omega|Q/c). \tag{44}$$

The transform F''_ω of the short pulse is seen to be $(i\omega/|\omega|)F'_\omega$ and so both the angular spectrum of the short pulse $v''_\omega(\mathbf{r})$ and its average V''_ω over the large sphere are the same as for the long pulse. Since the dual TM solutions clearly have the same energy spectra in the far field as the TE solutions, each of the quartet of focused-doughnut solutions, for given q_1 and q_2, has the same energy spectrum (both angular and average). We will write, in every case,

$$v'_\omega=v''_\omega=v_\omega,\quad V'_\omega=V''_\omega=V_\omega,\quad U_+=U_-=U_{\rm EM}. \tag{45}$$

Substituting (44) into (40) gives

$$v_\omega=\frac{\mu_0 f_0^2\pi^2\omega^4\sin^2\psi}{2r^2c^5}\exp(-2\omega Q/c). \tag{46}$$

Recall that the energy spectra are defined in (37)–(40) for positive frequency ω only. Integrating (46) over all frequency shows that the outgoing energy flux in the far field depends only on the polar angle ψ and is proportional to

$$[Q(\psi)]^{-5}\sin^2\psi. \tag{47}$$

This function depicts a diffracting doughnut centered on the z axis. The diffraction angle is much less than unity when

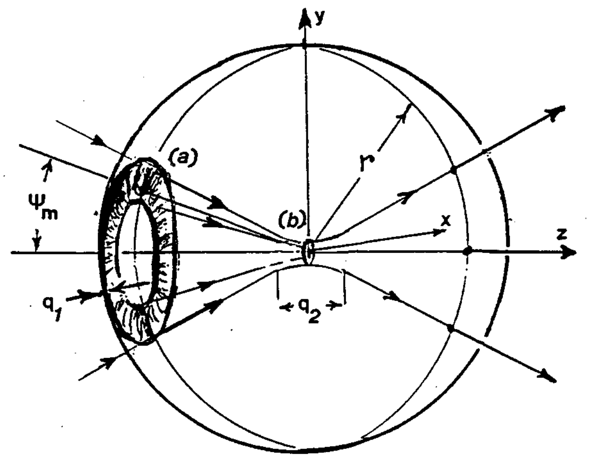

FIG. 1. Schematic of ''focused doughnut'' pulse (a) at time $t=-r/c$ and (b) at time zero of maximum focusing. The origin of the space coordinates is placed at the center of the focal region. The pulse thickness q_1 is also the nominal wavelength. The Rayleigh length q_2 of the focal region is also indicated. The divergence half angle ψ_m of the cone traced out by the focusing doughnut is nearly $\sqrt{q_1/q_2}$, similarly as for monochromatic Gaussian beams.

$q_2 \gg q_1$, in which case the angle ψ_m at which the energy flux is maximum is seen from (47) and (42) to be

$$\psi_m \to \sqrt{q_1/q_2}. \tag{48}$$

Numerical plots show the focused doughnut pulse to have a nominal wavelength of q_1 and the depth of its focal region is q_2. (See Fig. 1). Therefore the far-field doughnut diffraction angle (48) is essentially the same as for a monochromatic focused Gaussian pulse of wavelength q_1 and Rayleigh length q_2.

The energy spectrum V_ω itself is easily found from integrating (46) in (35):

$$V_\omega = 4\pi^3 c^{-5} \mu_0 f_0^2 \omega^4 \left[\frac{\cosh a}{a^2} - \frac{\sinh a}{a^3} \right] \exp[-\omega(q_1+q_2)/c], \tag{49}$$

where $a \equiv (q_2 - q_1)\omega/c$. This expression in (34) reproduces the result (33) for the total pulse energy.

With the expression (49), it is simple to calculate the cutoff frequency ω_η of the pulse spectrum, defined as the (angular) frequency above which the energy spectrum contains the fraction η of the total energy:

$$\eta = \frac{\int_{\omega_\eta}^{\infty} d\omega V_\omega}{\int_0^{\infty} d\omega V_\omega}. \tag{50}$$

When $q_2 \gg q_1$ (weak focusing) this gives

$$\eta = \exp(-\gamma_1)(1+\gamma_1+\tfrac{1}{2}\gamma_1^2) + O(q_1/q_2), \tag{51}$$

where $\gamma_1 = 2\omega_\eta q_1/c$. From this we find, for example, that the cutoff frequency is given by $\gamma_1 \to 5.322$ for $\eta = 0.1$ and by $\gamma_1 \to 8.406$ for $\eta = 0.01$ in the limit of weak focusing.

The exotic ''maximum focus'' limit $q_1 = q_2$ has a broad torus-shaped field imploding from the sides for negative times and exploding outward at positive times and always with its center of gravity at the origin. In this case

$$\eta = \exp(-\gamma_1)(1+\gamma_1+\tfrac{1}{2}\gamma_1^2+\tfrac{1}{6}\gamma_1^3+\tfrac{1}{24}\gamma_1^4). \tag{52}$$

From this we find, for example, that the cutoff frequency is given by $\gamma_1 \to 7.993$ for $\eta = 0.1$ and $\gamma_1 \to 11.605$ for $\eta = 0.01$. If we define the cutoff wavelength λ_m to be $2\pi c/\omega_\eta$ for $\eta = 99\%$, then the TE short form of this maximum focus ($q_2 = q_1$) pulse has its maximum electric field strength E_m at $t=0$ (the moment of maximum focus) within a few percent of $(U/\epsilon_0 \lambda_m^3)^{1/2}$. This is as one expects if the focused field is localized within a volume λ_m^3.

The short TE z-directed focused doughnut pulse has its maximum field strength E_m, expressed in terms of $\lambda_m = 2\pi c/\omega_\eta$ for $\eta = 99\%$, very near $[0.77 q_1/(\epsilon_0 q_2 \lambda_m^3)]^{1/2}$. This is as one would expect if the focused field is localized within a volume given by the [pulse length × (pulse diameter)^2].

V. ACCELERATION OF A RELATIVISTIC ELECTRON

In this section we calculate the energy W given to an electron traveling at the velocity c on axis ($\rho = 0$) with a short TM pulse, as shown in Fig. 2. The electron is decoupled from the pulse at $t = z = 0$ (maximum focus) by a conducting membrane that reflects the pulse and transmits the electron. See Fig. 2. Without this decoupling the energy gained by the electron up to the origin would be exactly returned to the field as its position z approached infinity. The short TM pulse has a z-directed electric field, and no magnetic field, at the position $z = ct$ of the electron. The short pulse imparts more energy W for a given field energy U and cutoff frequency ω_η than does the long TM pulse, so we calculate W for the short pulse only. In terms of the field $E_z''(0,0,z,t)$ experienced by the electron, the energy given to the electron is

$$W = e \int_{-\infty}^{0} dz E_z''[0,0,z,t(z)]. \tag{53}$$

Substituting (6) in (12) and $t = z/c$, we have, for the short TM pulse,

$$E_z'' = \frac{-16\mu_0 f_0 c q_2}{q_1^2} \frac{z}{(q_2^2+4z^2)^2}. \tag{54}$$

Substituting (54) in (53)

$$W = \frac{2ef_0}{q_1^2 q_2} \sqrt{\frac{\mu_0}{\epsilon_0}}. \tag{55}$$

It is useful to express this result in terms of (i) the energy $U = \mu_0 f_0^2 \pi^2 (q_1+q_2)/(4q_1^3 q_2^3)$ of the electromagnetic pulse and (ii) the Coulomb energy U_{Coul} of two electrons separated by $q_1(1+q_1/q_2)$ (i.e., nearly one cutoff wavelength):

$$U_{\mathrm{Coul}} \equiv \frac{e^2}{4\pi\epsilon_0 q_1[1+(q_1/q_2)]}. \tag{56}$$

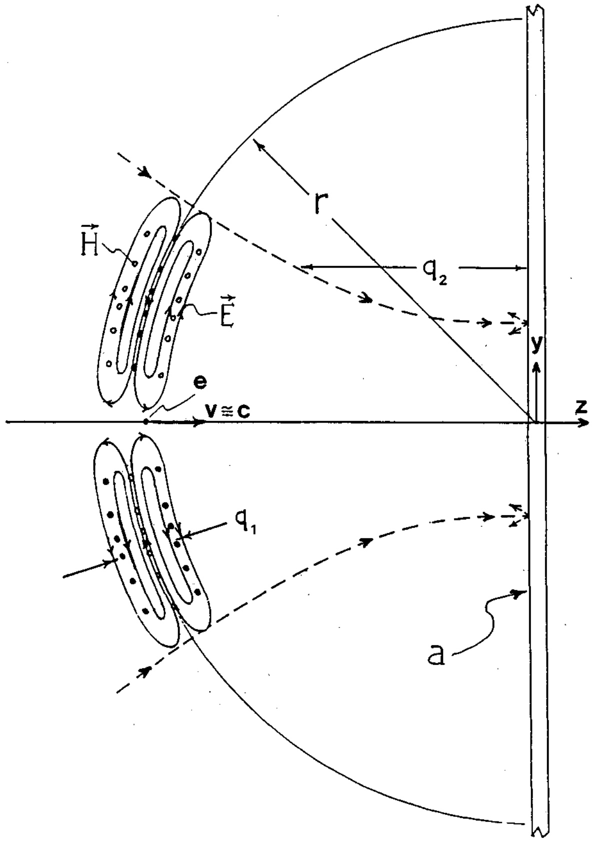

FIG. 2. Schematic of the electric $\vec{E}$ and magnetic $\vec{H}$ fields of a short TM focused doughnut pulse copropagating with an electron (e) moving in an axial trajectory. Open (closed) circles represent magnetic field lines out of (into) the plane of the diagram. The picture is drawn at time r/c before the maximum focus, which occurs at the origin of coordinates. The characteristic length parameters q_1 and q_2 are indicated. To achieve maximum acceleration of an electron of velocity c, in the axial trajectory, a metal film (a) should be placed in the $z=0$ plane to reflect the pulse and transmit the electron.

Then (55) can be rewritten

$$W=8\pi^{-1/2}\sqrt{UU_{\text{Coul}}}. \quad (57)$$

This form can be used as a scaling rule with the reference value, which we call the ‘‘one-one-one’’ rule: a 1-J (short TM) pulse whose (90%) cutoff wavelength is 1-μm can impart nearly 1 GeV of energy to a relativistic electron (or other unit-charged particle).

To be more precise, consider the weak focusing case ($q_2 \gg q_1$) for which we found in Sec. IV that 90% of the energy was at wavelengths longer than λ_m ($\lambda_m = 4\pi q_1/\gamma_1$ with $\gamma_1 = 5.32$). Then, if we take $\lambda_m = 1$ μm and the pulse energy U to be 1 J, (57) gives $W = 0.66$ GeV.

If the electron velocity is a constant v slightly less than c, then the integral (53) is easily redone to show that the energy W imparted to the electron is reduced by more than half once v/c is less than $2\psi_m^2$. Recall that the far-field divergence angle ψ_m was nearly $\sqrt{q_1/q_2}$ for the weak focusing case. We have assumed that the polarity of the short TM solution (set by the sign of f_0) was such as to accelerate the electron. Of course a solution of opposite polarity would cause the electron to lose the same energy W that we have calculated.

VI. CONCLUSION

We have examined a quartet of simple analytic finite-energy solutions of Maxwell's equation in vacuum. These constitute a small subset of the MPS solutions discovered by Ziolkowski. These solutions depict focused electromagnetic pulses that we call short (one-cycle) and long ($1\frac{1}{2}$-cycle) pulses of TM and TE character. These solutions are functions of a length parameter q_1 (essentially the wavelength) and another length parameter q_2 (essentially the depth of the focused region). We have assumed $q_2 \gg q_1$. We have found the far-field divergence angle ψ_m of these pulses to have essentially the same relation to q_1, when it is much less than q_2, as for common monochromatic Gaussian beams: $\psi_m \sim \sqrt{q_1/q_2}$. We have found the energy spectra of all four solutions to be the same (for given q_1 and q_2) and to be a simple analytic function of frequency ω, q_1, and q_2. A relativistic electron that is copropagating with the short TM pulse on its axis is shown to gain an energy W that is proportional to $[(\text{pulse energy})/(\text{cutoff wavelength})]^{1/2}$. We derived the one-one-one rule: W is nearly 1 GeV for a 1-J pulse having a cutoff wavelength of 1 μm. This energy is reduced if the electron velocity is less than c. However, the reduction is slight for highly focused TM pulses for which the effective interaction length of the pulse and electron is only a few wavelengths. At maximum focus, the energy density of the short (TM and TE) pulses decreases as r^{-10} at large distance r, while the energy density of the long pulses decreases as r^{-8} at maximum focus.

A great many questions concerning these focused one-cycle electromagnetic pulse solutions of Maxwell's equations remain. Is there is a simple transformation by which the long pulse solution can be obtained from the short pulse? What happens when the Hertz potential is x directed instead of z directed in deriving the fields? How are electrons accelerated if they intersect a solution along a path different from the one we have considered here? How can these solutions be generated, at least approximately, in the laboratory? These and many other such questions are under current investigation.

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